An Elementary "Proof" of Goldbach's Conjecture

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"*It's by talking nonsense that one gets to the truth! I talk nonsense, therefore I'm human.*"
Fyodor Dostoyevsky, **Notes from the Underground**

"*Every true creation begins with being an imitation.*" **Aristotle**

Introduction

It seems nonsense that somebody claims that he has a "proof" of a conjecture that has resisted without a solution, for more than two hundred and fifty years. Great mathematical minds have fought against this difficult problem and in the interim, they have created beautiful number theories. A sample of these efforts can be appreciated in the paper of Harald Andrés Helfgott[1] about the ternary Goldbach conjecture which states that *every odd number* $n \geq 7$ *is the sum of three primes*.

Here we present an algorithmic approach to the other version of Goldbach conjecture, i.e. that: *every even integer greater than 2 can be expressed as the sum of two primes*. The reason to use the term elementary "proof" is because this presentation is thought for school teachers interested to carry on at classroom, classical themes like the fundamental theorem of arithmetic or famous and historical important conjectures in number theory, such as, *Goldbach's* conjecture and the *Bertrand's postulate*.

Prior to drafting a proof we describe some preliminaries in order to be acquainted with the language we will use along this article.

In another paper[2] I introduce a new definition of natural numbers. We use those ideas here for approaching to a visualization of Goldbach's Conjecture.

Numerical Language

To begin with, let us associate to the natural number $\boldsymbol{a}$ [written in digital decimal form as

$\boldsymbol{a} = (a_n a_{n-1} \ldots a_1 a_0)$], the following n-th degree polynomial:

$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{i=n} a_{n-i} x^{n-i}$, with $a_i \in \{0, 1, 2, \ldots, 9\}$.

We will call polynomial $P_a(x) = \sum_{i=0}^{i=n} a_{n-i} x^{n-i}$, the *standard polynomial* associated to $\boldsymbol{a}$.

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[1] HELFGOTT, H. A. *Major Arcs for Goldbach's Problem*. https://arxiv.org/pdf/1305.2897v4.pdf.

[2] http://matematicasyfilosofiaenelaula.info/articulos/A_NEW_DEFINITION_OF_NATURAL_NUMBER%20_(%20first%20draft%20).pdf

Definition 1

We define the natural number $\boldsymbol{a} = (a_n a_{n-1} \ldots a_1 a_0)$, as the family $\mathscr{F}$, of all equivalent polynomials module a. Symbolically

$\Omega_a = \{ \sum_{j=0}^{j=n} b_{n-j} x^{n-j} \in \mathbb{N}[x]$, such that, any pair of polynomials P and Q in the family $\mathscr{F}$, satisfies: $P(10) = Q(10) = \boldsymbol{a}\}$.

We mean by $\mathbb{N}[x]$ the set of all one variable polynomials with coefficients in the set $\mathbb{N}$ of natural numbers.

Definition 2

A number $\boldsymbol{a}$, defined using Ω_a, is said to be *composite* if there exists a polynomial P in Ω_a, which can be factored. Otherwise a will be called *prime*.

Note that, we do not mention division at all, since finding prime numbers is an algorithmic process, as I have shown in other papers related with this topic. Right here I present a pair of examples, to show how, the process goes on.

Example 1

Find the prime factors of $\boldsymbol{n} = 1872$.

First of all we make explicit the family Ω_n that define 1872, using the syntactic property[3], $x^n = 10x^{n-1}$. Some of the polynomials are shown in the following list:

$$\Omega_n = \{x^3 + 8x^2 + 7x + 2,\ 18x^2 + 7x + 2,\ 17x^2 + 17x + 2,\ 16x^2 + 27x + 2,\ \ldots,\ 187x + 2,\ 1872\}$$

The defining characteristic of all these polynomials is that, they are congruent module 1872, i. e., given any P and Q in the set, P(10) = Q(10) = 1872. When we are working inside the set, we see all its elements as equals and we make the following changes licitly:

$$18x^2 + 7x + 2 = 18x^2 + 6x + 12 = 2(9x^2 + 3x + 6) = 2(8x^2 + 13x + 6) = 2(8x^2 + 12x +16) = 2^2(4x^2 + 6x + 8) = 2^3(2x^2 + 3x + 4) = 2^3(2x^2 + 2x + 14) = 2^4(x^2 + x + 7) = 2^4(10x + x + 7) = 2^4(11x + 7) = 2^4(9x + 2x + 7) = 2^4(9x + 27) = 2^4 \cdot 3(3x + 9) = 2^4 \cdot 3^2(x + 3) = 2^4 \cdot 3^2 \cdot 13 = 1872.$$

In the precedent algorithmic process, we tried to go step by step. We can, of course, make some shortcuts to simplify the process. The remarkable thing here is that, you start the process, with a number $\boldsymbol{n}$ and you finish it, with its unique prime representation, without none exterior operations or mysterious processes. This experience is like the realization of a dream that started with Euler, namely, to find a polynomial that gives up only prime numbers. All this stuff is connatural to the definition of number $\boldsymbol{n}$.

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[3]See one of my first articles in this topic: *Beginning Abstract Algebra at Elementary School*, at: http://www.matematicasyfilosofiaenelaula.info/articulos/Beginning_Abstract_Algebra_at_Elementary_School.pdf

We may emphasize that our new definition of natural number implies directly the so called, *fundamental theorem of arithmetic*. The fundamental theorem of arithmetic states that, every positive integer (except the number 1) can be represented in exactly one way, apart from rearrangement, as a product of one or more primes.

Example 2

Determine whether $\boldsymbol{n} = 2017$ is prime or composite.

As before, we work inside Ω_n, to perform the appropriate transformations to decide the primality of 2017. The standard polynomial associated to 2017 is $2x^3 + x + 7$.

$2x^3 + x + 7 = 20x^2 + x + 7 = 19x^2 + 11x + 7 = 18x^2 + 21x + 7 = 17x^2 + 31x + 7 = 16x^2 + 41x + 7 = 15x^2 + 51x + 7 = 14x^2 + 61x + 7 = 13x^2 + 71x + 7 = 12x^2 + 81x + 7 = 12x^2 + 79x + 27 = 12x^2 + 79x + 3\cdot 9 = 11x^2 + 89x + 3\cdot 9 = 10x^2 + 99x + 3\cdot 9$.

None of these polynomials can be factored as product of linear polynomials with coefficients in $\mathbb{N}$. Therefore, 2017 is a prime number.

In the precedent process we finish at the $10x^2$ level, since we are looking for linear factors, in such a way that the product of the quadratic coefficient (or its factors) and the independent term (or its factors) reproduce the coefficient of x in the quadratic polynomial; from this point on, the coefficient of the middle part start to be greater than that product. See my articles about factoring at http://matematicasyfilosofiaenelaula.info [4].

The Goldbach's Conjecture

Every even integer greater than 2, can be expressed as the sum of two primes.

An Elementary "Proof"

Let $\boldsymbol{n} = 2\boldsymbol{a}$. If $\boldsymbol{a}$ is prime, $2\boldsymbol{a}$ obviously can be expressed as the sum of two primes: $\boldsymbol{n} = 2\boldsymbol{a} = \boldsymbol{a} + \boldsymbol{a}$ and so $\boldsymbol{n}$ is the sum of two primes.

Now, suppose $\boldsymbol{a}$ is composite. For this case, according definition 2, inside Ω_a there exists a polynomial P, which factors as $P(x) = P_1(x)P_2(x)\ldots P_k(x)$. Then $2\boldsymbol{a} = 2P(10) = 2P_1(10)P_2(10)\ldots P_k(10) = 2p_1p_2\ldots p_k$, where all p_i, $1 \le i \le k$, are prime numbers ordered from lesser to major. This means that $\boldsymbol{n} = 2p_1p_2\ldots p_k$.

Using the algorithm to check primality described in my paper *Roots and Stems*[5], we can find an odd prime $p > p_1$, and different from p_i, $1 \le i \le k$, such that $\boldsymbol{n} - p$ be prime. Through this procedure, we can express $\boldsymbol{n}$ as a sum of two primes, namely: $\boldsymbol{n} = p + (\boldsymbol{n} - p)$. This completes the proof.

Let's see a couple of examples.

Example 1

[4] For instance: http://matematicasyfilosofiaenelaula.info/articulos/Numbers_as_a_product_of_Primes.pdf

[5] http://matematicasyfilosofiaenelaula.info/articulos/ROOTS%20AND%20STEMS%20V.1.pdf

Take $\boldsymbol{n}$ = 746 and find its prime representation using Ω_n.

$746 = 7x^2 + 4x + 6 = 6x^2 + 14x + 6 = 2(3x^2 + 7x + 3) = 2\cdot373$. Since, 373 is prime, 746 can be represented as the sum of two primes, namely, 373 + 373.

373 is prime number, not just because it appears in a prime table, but more importantly, because, according the above definition, none of the following equivalent polynomials module 373 are factorizable.

$3x^2 + 7x + 3 = 2x^2 + 17x + 3 = x^2 + 27x + 3 = x^2 + 21x + 7\cdot9$.

Example 2

If $\boldsymbol{n}$ = 8262 and $\boldsymbol{a}$ = 4131, we can first check $\boldsymbol{a}$ for primality and, then we factor $\boldsymbol{n}$ in Ω_n, using its standard polynomial

$4131 = 4x^3 + x^2 + 3x + 1 = 3x^3 + 11x^2 + 3x + 1 = 3x^3 + 9x^2 + 23x + 1 = 3x^3 + 9x^2 + 21x + 21 = 3(x^3 + 3x^2 + 7x + 7) = 3(13x^2 + 7x + 7) = 3(12x^2 + 17x + 7) = 3(12x^2 + 15x + 27) = 3^2(4x^2 + 5x + 9) = 3^2(3x^2 + 15x + 9) = 3^3(x^2 + 5x + 3) = 3^3(15x + 3) = 3^4(5x + 1) = 3^4(3x + 21) = 3^5(x + 7) = 3^5\cdot17$.

Then $\boldsymbol{n} = 2\boldsymbol{a} = 2\cdot3^5\cdot17$.

The first prime p such that, $8262 - p$ is prime is 29. So, $8262 = (8262 - p) + p = 8233 + 29$.

As a matter of fact, in Ω_n we have:

$8233 = 8x^3 + 2x^2 + 3x + 3 = 82x^2 + 3x + 3 = 81x^2 + 13x + 3 = \ldots = 41x^2 + 407x + 7\cdot9$.

No one of the above polynomials factor in $\mathbf{Z}[x]$ and so, 8233 is a prime, according definition 2, above.

The precedent ideas can be reduced to the following "theorem"

Theorem

If $\boldsymbol{a} = (a_n a_{n-1} \ldots a_1 a_0) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \sum_{i=0}^{i=n} a_{n-i}x^{n-i}$, with $x = 10$ and $a_i \in \{0, 1, 2, ..., 9\}$, *is an even positive number greater than or equal to 4*, then $\boldsymbol{a}$ *can be expressed as a sum of two prime numbers, of the type:* $(\boldsymbol{a} - 2) + 2$, $(\boldsymbol{a} - 3) + 3$, $(\boldsymbol{a} - 5) + 5$, …, $(\boldsymbol{a} - p_k) + p_k = \boldsymbol{a}$, where p_k *is prime and*, $\boldsymbol{a}/2 \leq p_k < \boldsymbol{a}$.

Proof

If $\boldsymbol{a}/2$ is a prime, the conclusion is immediate. Let us suppose that $\boldsymbol{a}/2$ is composite. Then $2(\boldsymbol{a}/2) = \boldsymbol{a}$, is also composite and for definition $\boldsymbol{a} = p_1 p_2 \ldots p_i$, where $p_1 = 2$ and p_i is the major of all the prime factors.

We choose primes $p > 2$ different from p_j, $1 \leq j \leq i$, and test $\boldsymbol{a} - p$ for primality. Whenever we find a prime like this, we get a representation of $\boldsymbol{a}$ as a sum of two primes.

Example

Let $\boldsymbol{a} = 2360$ and let us try to find a representation of $\boldsymbol{a}$ as a sum of two primes.

Again, Ω_a is the class defining $\boldsymbol{a}$. Inside this class we can perform the following operations:

$2x^3 + 3x^2 + 6x = x(2x^2 + 3x + 6) = x(2x^2 + 2x + 16) = 2x(x^2 + x + 8) = 2x(11x + 8) = 2x(10x + 18) = 2^2x(5x + 9) = 2^2 \cdot 10(5 \cdot 10 + 9) = 2^3 \cdot 5 \cdot 59$.

We can choose $p = 3$ and check whether $2360 - 3 = 2357$ is a prime number. Following the same assumptions as before, we get:

$2357 = 2x^3 + 3x^2 + 5x + 7 = 23x^2 + 5x + 7 = 22x^2 + 15x + 7 = 21x^2 + 25x + 7 = \ldots = 12x^2 + 113x + 3 \cdot 9$.

Since the above polynomials are not factorizable in $\mathbf{Z}[x]$, we conclude that 2357 is a prime number. This means that 2360 is the sum of two primes: 2360 = 2357 + 3.

We could check other options taking $p \neq 2, 5, 59$; like 7, 11, 13, etc.

Theorem (Bertrand's Postulate): *For every integer* $\boldsymbol{a} \geq 2$, *there is a prime p satisfying*: $\boldsymbol{a} < p < \boldsymbol{2a}$.

Proof

Given number $2\boldsymbol{a}$, Goldbach's Conjecture warranties that there exist primes p and q such that $p + q = 2\boldsymbol{a}$. Clearly both, p and q are lesser than $2\boldsymbol{a}$. That means that one of them is greater than $\boldsymbol{a}$ and the other one is lesser than $\boldsymbol{a}$**.** Therefore there is a prime between $\boldsymbol{a}$ and $2\boldsymbol{a}$.

We could also use the same technique as in the proof of Goldbach's Conjecture.

Let since $2\boldsymbol{a}$ is composite, there is a polynomial $P_{2a}(x)$ in Ω_{2a}, which factorize:

$P_{2a}(x) = 2P_a(x) = 2P_1(x)P_2(x)\ldots P_k(x)$. So

$2\boldsymbol{a} = 2P_1(10)P_2(10)\ldots P_k(10) = 2p_1p_2\ldots p_k$, where all p_i, $1 \leq i \leq k$, are prime numbers ordered from lesser to major.

We can test for primality the number $2\boldsymbol{a} - p$, where p is an odd prime and, $p \neq p_i$, $1 \leq i \leq k$, till $2\boldsymbol{a} - p$ be prime. This prime satisfies the conditions of the theorem.

Example

Let us find a prime number between 3591 and 7182.

We start with the prime factorization of $2\boldsymbol{a} = 7182$ inside of Ω_{2a}.

$7x^3 + x^2 + 8x + 2 = 6x^3 + 11x^2 + 8x + 2 = 6x^3 + 10x^2 + 18x + 2 = 2(3x^3 + 5x^2 + 9x + 1) = 2(35x^2 + 9x + 1) = 2(34x^2 + 19x + 1) = 2(33x^2 + 29x + 1) = 2(32x^2 + 39x + 1) = \ldots = 2(17x^2 + 189x + 1) = 2(17x^2 + 181x + 9\cdot 9)$.

None of the above polynomials inside the parenthesis are factorizable; so, 3591 is a prime. Then, let's look for a prime p, such that, $2\boldsymbol{a} - p$, be also a prime.

We do not take $p = 2$, since $2\boldsymbol{a} - 2 = 2(\boldsymbol{a} - 1)$ is not prime. So let's check: $2\boldsymbol{a} - 3$, $2\boldsymbol{a} - 5$, etc. Just at, $2\boldsymbol{a} - 5 = 7182 - 5 = 7177$, the process finish, since, the following polynomials are not factorizable in $\mathbf{Z}[x]$ the ring of polynomials with integer coefficients.

$\{7x^3 + x^2 + 7x + 7, 71x^2 + 7x + 7, 70x^2 + 17x + 7, \ldots, 36x^2 + 357x + 7, 36x^2 + 355x + 3\cdot 7\}$.

In conclusion, 7177 is a prime between 3591 and 7182.

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